Arndt Compositions: Connections with Fibonacci Numbers, Statistics, and Generalizations

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Arndt Compositions

A *composition* of an integer n is a sequence $(x_1, \ldots, x_k)$ of positive integers that satisfies $\sum_{i=1}^{k} x_i = n$. We say that a composition is an *Arndt composition*[a] if it also satisfies $\boldsymbol{x_{2i-1} > x_{2i}}$ for all $\boldsymbol{i > 0}$.

When the length—the number of summands—of the composition is odd, this condition on the last summand is vacuously true. We denote by $\mathcal{A}_n$ the set of Arndt compositions that exist for an integer $n > 0$, $a_n := |\mathcal{A}_n|$, and $\mathcal{A} = \bigcup_{n \geq 1} \mathcal{A}_n$.

For example,

$$6 + 3 + 2 + 1 + 7 + 4 + 5$$

is a composition in $\mathcal{A}_{28}$. Table 1 shows some examples and the first values of a_n, which seem to coincide with the Fibonacci number sequence.

n	$\mathcal{A}_n$	a_n
1	1	1
2	2	1
3	3, 21	2
4	4, 31, 211	3
5	5, 41, 32, 311, 212	5
6	6, 51, 42, 411, 321, 312, 213, 2121	8
7	7, 61, 52, 511, 43, 421, 412, 322, 313, 3121, 214, 2131, 21211	13

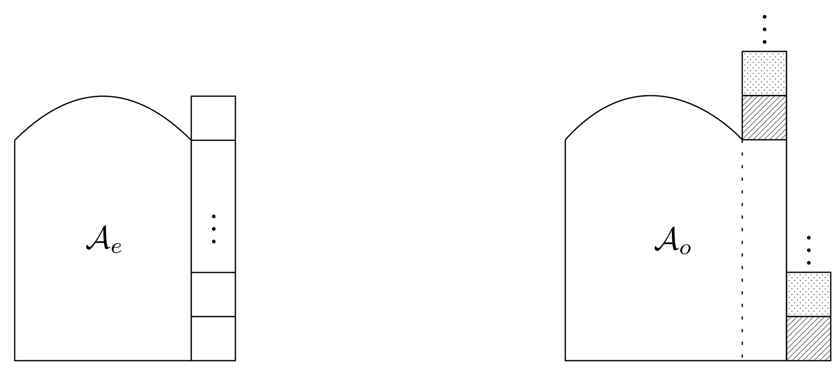
Table 1. Arndt compositions up to $n = 7$. [2]

Hopkins and Tangboonduangjit [2] published two combinatorial proofs that verify this observation, one of which consists of an explicit bijection with the number of compositions whose parts are 1 or 2. Our objective is to provide another proof using generating functions and to study some statistics and generalizations regarding $\mathcal{A}$.

Counting Sequence

Let $A(z) = \sum_{n \geq 0} a_n z^n$ be the generating function of $\mathcal{A}$, and let $A_o(z)$ and $A_e(z)$ be the respective generating functions of those Arndt compositions of odd and even length, denoted by $\mathcal{A}_o$ and $\mathcal{A}_e$.

By representing the compositions as bar graphs, the following reasoning is natural. Every composition in $\mathcal{A}_o$ of length greater than one is an arbitrary summand concatenated with a composition in $\mathcal{A}_e$ —Figure 1a—. Every composition in $\mathcal{A}_e$ is a composition in $\mathcal{A}_o$ concatenated with pairs of units, with one unit in the penultimate part and the other in the last part —Figure 1b—.



(a) Composition in $\mathcal{A}_o$. (b) Composition in $\mathcal{A}_e$.

Figure 1. Structure of compositions in $\mathcal{A}$.

Therefore, we have the functional system

$$A(z) = A_o(z) + A_e(z), \qquad A_o(z) = \frac{z}{1-z} + A_e(z)\frac{z}{1-z}, \qquad A_e(z) = A_o(z)\frac{z^2}{1-z^2}.$$

Whose solution is

$$A(z) = \frac{z}{1-z-z^2}, \qquad A_e(z) = z^2 A(z), \qquad A_o(z) = z + zA(z).$$

Fibonacci Numbers!

This generating function is the same as the Fibonacci numbers sequence! We can conclude that $a_n = F_n$. Not only that, for each $n \geq 2$, there are F_{n-1} Arndt compositions of odd length and F_{n-2} of even length.

What happens when $x_{2i-1} \geq k + x_{2i}$?

If we modify the original restriction to $x_{2i-1} \geq k + x_{2i}$ with $k \in \mathbb{Z}$, we obtain

$$A^{(k)}(z) = \begin{cases} \dfrac{z - z^3 + z^{k+2}}{1 - z - z^2 + z^3 - z^{k+2}}, & \text{if } k \geq 0 \\[2ex] \dfrac{z + z^2 - z^{-k+3}}{1 - z - 2z^2 + z^{-k+3}}, & \text{if } k < 0. \end{cases}$$

Thus, $a_n^{(k)} = a_{n-1}^{(k)} + a_{n-2}^{(k)} - a_{n-3}^{(k)} + a_{n-k-2}^{(k)}$, for all $n > k+2$ and $k > 0$. For $k < 0$ and $n > -k+3$, $a_n^{(k)} = a_{n-1}^{(k)} + 2a_{n-2}^{(k)} - a_{n+k-3}^{(k)}$.

What if $|x_{2i-1} - x_{2i}| \geq k$?

If $|x_{2i-1} - x_{2i}| \geq k$ for $k > 0$, we obtain the expression

$$A^{(k)}(z) = \frac{z - z^3 + 2z^{k+2}}{1 - z - z^2 + z^3 - 2z^{k+2}},$$

from which $a_n^{(k)} = a_{n-1}^{(k)} + a_{n-2}^{(k)} - a_{n-3}^{(k)} + 2a_{n-k-2}^{(k)}$. In the case $k = 1$, the counting sequence can be expressed in terms of the Tribonacci numbers.

[a] These compositions are named in honor of Joerg Arndt, who initially studied them and published in 2013 in the *Online Encyclopedia of Integer Sequences* (A000045) that for each $n > 0$, a_n coincides with the n-th Fibonacci number F_n, without providing a formal proof.

Statistics

Using this method is advantageous, as it allows us to study different statistics and recurrence relations that are not immediately apparent when using purely combinatorial arguments. Below are some of the most interesting ones.

Number of Summands

If $B^{(k)}(z)$ is the generating function of the compositions in $\mathcal{A}$ such that their length is $k > 0$, it can be deduced that

$$B^{(k)}(z) = \frac{z^{k+\lfloor k/2 \rfloor}}{(1-z)^k (1+z)^{\lfloor k/2 \rfloor}}.$$

By extracting the n-th coefficient of this expression and summing over all $k \geq 1$, we recover the n-th Fibonacci number.

$$F_n = \sum_{k=1}^{\lfloor \frac{2n+1}{3} \rfloor} \sum_{i=\lfloor k/2 \rfloor}^{n-k} \binom{n-i-1}{k-1} \binom{i-1}{\lfloor k/2 \rfloor - 1} (-1)^{i+\lfloor k/2 \rfloor}. \qquad (n \geq 1)$$

Using similar methods, we have studied the size of the first and last summands, the size of the largest and smallest summands, and the semiperimeter and number of interior points associated with the bar graph of each Arndt composition.

Size of the Last and First Summands

Among the results found for these statistics, a new combinatorial interpretation of two sequences stands out. We denote by $c_n^{(k)}$ the number of Arndt compositions whose last summand is k, and similarly, by $d_n^{(k)}$ those whose first summand is k.

The sum of the last summands over all Arndt compositions of n is **OEIS A014217** and the sum of all the first summands is **OEIS A129696**.

$$\sum_{k=1}^{n} k c_n^{(k)} = \left\lfloor \left(\frac{1+\sqrt{5}}{2}\right)^n \right\rfloor, \qquad \sum_{k=1}^{n} k d_n^{(k)} = F_{n+3} - \lfloor n/2 \rfloor - 2.$$

Number of Interior Points and Semiperimeter

When we represent these compositions using bar graphs, it can be interesting to count the number of interior points and the semiperimeter.

We denote by $A(z,p,q)$ the generating function of the compositions in $\mathcal{A}$ whose last summand is k and the variables p, q respectively keep track of the semiperimeter and the number of interior points. It can be proven that (see [1])

$$A(z,p,q) = \det \begin{bmatrix} \alpha_1 & 1+\beta_1+\gamma_1 \\ \alpha_2 & 1+\beta_2+\gamma_2 \end{bmatrix} (z,p,q) \Big/ \det \begin{bmatrix} \beta_1 & 1+\gamma_1 \\ 1+\beta_2 & \gamma_2 \end{bmatrix} (z,p,q),$$

where

$$\alpha_1(z,p,q) = \sum_{n \geq 1} \frac{p^{2n+1}(q-p)^{n-1} q^{(n-1)^2} z^{n(n+1)}}{(q^2z^2; q^2z^2)_{n-1} (qz, pqz^2; q^2z^2)_n}, \qquad \beta_1(z,p,q) = \sum_{n \geq 1} \frac{p^{2n-1}(q-p)^{n-1} q^{(n-1)^2} z^{n(n+1)-1}}{(q^2z^2, pqz^2; q^2z^2)_{n-1} (qz; q^2z^2)_n},$$

$$\gamma_1(z,p,q) = \sum_{n \geq 1} \frac{p^{2n}(q-p)^{n-1} q^{(n-1)^2} z^{n(n+1)}}{(pqz^2; q^2z^2)_{n-1} (qz; qz)_{2n}}, \qquad \alpha_2(z,p,q) = \sum_{n \geq 1} \frac{p^{2n}(q-p)^{n-1} q^{(n-1)(n-2)} z^{n^2}}{(pz; q^2z^2)_n (qz; qz)_{2n-2}},$$

$$\beta_2(z,p,q) = \sum_{n \geq 1} \frac{p^{2n}(q-p)^{n} q^{n(n-1)} z^{n(n+2)}}{(pz; q^2z^2)_n (qz; qz)_{2n}}, \qquad \gamma_2(z,p,q) = \sum_{n \geq 1} \frac{p^{2n-1}(q-p)^{n-1} q^{(n-1)(n-2)} z^{n^2}}{(qz; q^2z^2)_n (pz, q^2z^2; q^2z^2)_{n-1}}.$$

Hence,

$$\begin{aligned} A(z,p,q) = {} & p^2z + p^3z^2 + 2p^4z^3 + 3p^5z^4 + \left(4p^6 + p^5q\right) z^5 + \left(6p^7 + 2p^6q\right) z^6 + \left(9p^8 + 2p^7q + 2p^6q^2\right) z^7 \\ & + \left(13p^9 + 3p^8q + 5p^7q^2\right) z^8 + \left(19p^{10} + 5p^9q + 8p^8q^2 + 2p^7q^3\right) z^9 \\ & + \left(28p^{11} + 7p^{10}q + 14p^9q^2 + \mathbf{5p^8q^3} + p^7q^4\right) z^{10} + O\left(z^{11}\right). \end{aligned}$$

For example, the term $5p^8q^3z^{10}$ indicates that there are 5 Arndt compositions of 10 whose semiperimeter is 8 and with 3 interior points—Figure 2.

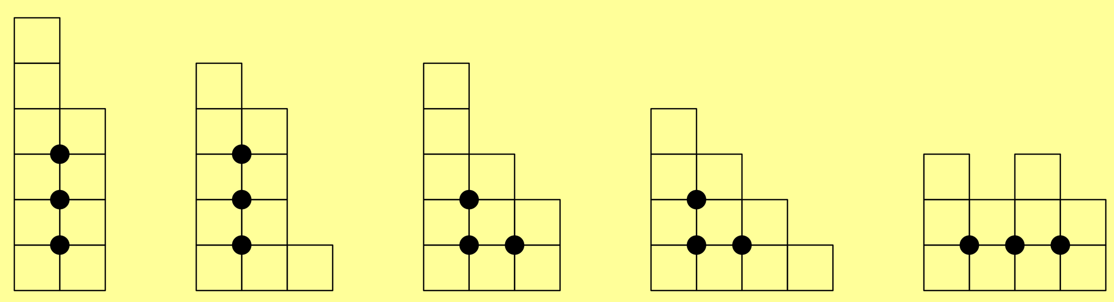


Figure 2. Interior points and semiperimeter.

This result is closely related to **Pick's Theorem**, which tells us that the area of any simple polygon $Q \subset \mathbb{R}$ with integer vertices is given by $A(Q) = I + \frac{B}{2} - 1$, where I and B are respectively the number of integer vertices in the interior and on the boundary of Q.

k-Block Arndt Compositions

What happens when, instead of comparing pairs of summands, we compare triples or quadruples? What happens when we consider k decreasing summands as blocks of Arndt compositions?

Let $\mathcal{W}^{(k)}$ be the class of—non-empty—compositions that satisfy $x_{ki-k+1} > x_{ki-k+2} > \cdots > x_{ki}$ for $i, k > 0$. These are called k-block Arndt compositions.

The g.f. of the number of k-block Arndt compositions is

$$W^{(k)}(z) = \frac{1}{1 - P_{\text{diff}}^{(k)}(z)} \sum_{j=1}^{k} P_{\text{diff}}^{(j)}(z) = \frac{\sum_{j=1}^{k} z^{\binom{j+1}{2}}/(z;z)_j}{1 - z^{\binom{k+1}{2}}/(z;z)_k}.$$

References

[1] Checa, Daniel F.: *Arndt Compositions: Connections with Fibonacci Numbers, Statistics, and Generalizations*, December 2023. `https://oeis.org/A000045/a000045_3.pdf`.

[2] Hopkins, Brian and Tangboonduangjit, Aram: *Verifying and Generalizing Arndt's Compositions*. The Fibonacci Quarterly, 60(5):181–186, December 2022, ISSN 0015-0517. `https://www.fq.math.ca/60-5.html`.